J. Math. Anal. Appl. 345 (2008) 410–419

Contents lists available at ScienceDirect

J. Math. Anal. Appl.

www.elsevier.com/locate/jmaa

Local rotundity structure of Cesàro–Orlicz sequence spaces

Paweł Foralewski [a], Henryk Hudzik [a,*], Alicja Szymaszkiewicz [b]

[a] Faculty of Mathematics and Computer Science, Adam Mickiewicz University, Umultowska 87, 61-614 Poznań, Poland
[b] Institute of Mathematics, Szczecin University of Technology, Al. Piastów 48/49, 70-310 Szczecin, Poland

ARTICLE INFO

Article history:
Received 23 February 2007
Available online 11 April 2008
Submitted by J. Bastero

Keywords:
Orlicz function
Orlicz sequence space
Cesàro–Orlicz sequence space
Extreme point
Strong U-point
Best approximation

ABSTRACT

Some criteria for extreme points and strong U-points in Cesàro–Orlicz spaces are given. In consequence we find a Cesàro–Orlicz sequence space different from c_0 which has no extreme points. Some examples show that in these spaces the notion of the strong U-point is essentially stronger than the notion of the extreme point. Various examples presented in this paper show that there are some differences between criteria for extreme points and strong U-points in Orlicz spaces and in Cesàro–Orlicz spaces. We also show that the uniqueness of the local best approximation needs the notion of SU-point, that is, the notion of the extreme point is not strong enough here.

1. Preliminaries

Let $(X, \|\cdot\|)$ be a real Banach space and let $B(X)$ and $S(X)$ be the closed unit ball and the unit sphere of X, respectively.

A point $x \in S(X)$ is called an extreme point of $B(X)$ if for every $y, z \in B(X)$ with $x = \frac{y+z}{2}$, we have $y = z$. The notion of extreme point plays an important role in some branches of mathematics. For example, the Krein–Milman theorem, Choquet integral representation theorem, Rainwater theorem on convergence in the weak topology, Bessaga–Pełczyński theorem and Elton test for unconditional convergence are formulated in terms of extreme points (see [12, Chapter IX]).

A point $x \in S(X)$ is said to be a strong U-point (SU-point for short) of $B(X)$ if for any $y \in S(X)$ with $\|x + y\| = 2$, we have $x = y$ (cf. [2], where SU-points are called rotund points). Recall that the nature of an SU-point is such that a point $x \in S(X)$ is a point of local uniform rotundity if and only if x is a point of compact local uniform rotundity and an SU-point (see [8]).

It is obvious that a Banach space X is rotund if and only if every point of $S(X)$ is an extreme point of $B(X)$, as well as if and only if any point of $S(X)$ is an SU-point of $B(X)$, but the notion of SU-point is essentially stronger than the notion of extreme point. Namely in l^2_∞ (two-dimensional l_∞ space) the points $x = (1, 1)$ and $y = (1, -1)$ are extreme points of $B(l^2_\infty)$. Since $x \neq y$ and $\|x + y\| = 2$, neither x nor y is a strong U-point of $B(l^2_\infty)$.

It is well known that rotundity of a normed space X is important for the uniqueness of the best approximation element in any bounded closed convex and nonempty set $A \subset X$ for any $x \in X \setminus A$. Namely, if X is a rotund normed space, A is a bounded closed convex and nonempty set in X and $x \in X \setminus A$, then if $y \in A$ is such that $\|x - y\| = d(x, A) := \inf\{\|x - z\|: z \in A\}$, then for any $z \in A$, $z \neq y$, we have $d(x, A) < \|x - z\|$. If X is not rotund, then there is a bounded closed convex and nonempty set A in X and $x \in X \setminus A$ such that there is a continuum of points in A that realize the distance $d(x, A)$.

* Corresponding author.
E-mail addresses: katon@amu.edu.pl (P. Foralewski), hudzik@amu.edu.pl (H. Hudzik), alicja.szymaszkiewicz@ps.pl (A. Szymaszkiewicz).

0022-247X/$ – see front matter
doi:10.1016/j.jmaa.2008.04.016

If we want to consider the problem of the uniqueness of the best approximation locally, that is, for some fixed $x \in X \setminus A$ only, then the notion of SU-point can be applied (see Note 1). Since in that note the notion of extreme point cannot be used in place of the SU-point (see Remark 1), so the notion of the SU-point is important for the local best approximation problem.

A map $\varphi : \mathbb{R} \to [0, +\infty]$ is said to be an Orlicz function if φ is even, convex, left continuous on $\mathbb{R}_+$, continuous at zero, $\varphi(0) = 0$ and $\varphi(u) \to \infty$ as $u \to \infty$ (see [4,18,21,23,24,26,27]). For any Orlicz function φ we denote

$$a_\varphi = \sup\{u \geqslant 0:\ \varphi(u) = 0\} \quad \text{and} \quad b_\varphi = \sup\{u \geqslant 0:\ \varphi(u) < \infty\}.$$

Given any Orlicz function φ, we define on l^0 (the space of all real sequences) the following convex modular $\mathrm{I}_\varphi : l^0 \to [0, \infty]$:

$$\mathrm{I}_\varphi(x) = \sum_{i=1}^{\infty} \varphi\big(x(i)\big).$$

The space

$$l_\varphi = \big\{x \in l^0:\ \mathrm{I}_\varphi(\lambda x) < \infty \text{ for some } \lambda > 0\big\}$$

is called the Orlicz sequence space (see [4,18,21,23,24,26,27]). We equip this space with the Luxemburg norm

$$|||x|||_\varphi = \inf\left\{\lambda > 0:\ \mathrm{I}_\varphi\left(\frac{x}{\lambda}\right) \leqslant 1\right\}.$$

The arithmetic mean map σ is defined on l^0 by the formula

$$\sigma x = \big(\sigma x(i)\big)_{i=1}^{\infty}, \quad \text{where } \sigma x(i) = \frac{1}{i}\sum_{j=1}^{i}\big|x(j)\big|$$

for any $i \in \mathbb{N}$ and $x = (x(i))_{i=1}^{\infty} \in l^0$. Given any Orlicz function φ, we define on l^0 another convex modular $\varrho_\varphi : l^0 \to [0, \infty]$, by

$$\varrho_\varphi(x) = I_\varphi\big(\sigma(x)\big)$$

and the Cesàro–Orlicz sequence space

$$\mathrm{ces}_\varphi = \big\{x \in l^0:\ \sigma x \in l_\varphi\big\}$$

(see [9,25]). We equip this space with the norm $\|x\|_\varphi = |||\sigma(x)|||_\varphi$. The Cesàro–Orlicz sequence spaces $\mathrm{ces}_\varphi = (\mathrm{ces}_\varphi, \|\cdot\|_\varphi)$ have the Fatou property. Consequently, ces_φ are Banach spaces (see [23]).

We also define a subspace $(\mathrm{ces}_\varphi)_a$ of ces_φ by the following formula:

$$(\mathrm{ces}_\varphi)_a = \left\{x \in \mathrm{ces}_\varphi:\ \forall k > 0\ \exists n_k \in \mathbb{N} \text{ such that } \sum_{n=n_k}^{\infty} \varphi\left(\frac{k}{n}\sum_{i=1}^{n}\big|x(i)\big|\right) < \infty\right\}.$$

The space $(\mathrm{ces}_\varphi)_a$ is a closed and separable subspace of ces_φ and $(\mathrm{ces}_\varphi)_a$ is the subspace of all order continuous elements of ces_φ. For the definition of order continuous elements in a Banach lattice we refer to [17] and [22].

In particular cases, when $\varphi(u) = |u|^p$ for $1 \leqslant p < \infty$ or $\varphi(u) = 0$ if $|u| \leqslant 1$ and $\varphi(u) = \infty$ if $|u| > 1$ (which corresponds to $p = \infty$), we get the well-known Cesàro sequence spaces ces_p and ces_∞. They appeared in 1968 as the problem of the Dutch Mathematical Society to find their duals (see [1, Problem 2]). A regular investigation of Cesàro sequence spaces was done in [28] (see also [3,16,20]). At the end of the previous century several authors studied some geometric properties of these spaces (see [5–7,10,11,19]).

In recent years the theory of Cesàro–Orlicz sequence spaces has been studied intensively. Some basic topological properties (nontriviality, order continuity, separability and relationships between the modular and the norm defined itself) as well as some geometric properties (Fatou property, strict monotonicity and rotundity) were considered in [9]. Recently Maligranda, Petrot and Suantai in their remarkable paper [25] calculated n-dimensional James constans in Cesàro and Cesàro–Orlicz sequence spaces. They concluded from this result that neither Cesàro sequence spaces ces_p for $1 < p \leqslant \infty$ nor Cesàro–Orlicz sequence spaces ces_φ generated by Orlicz functions φ satisfying condition δ_2 are uniformly nonsquare, and they are not even B-convex. They also calculated the James constant for two-dimensional space $\mathrm{ces}_2^{(2)}$.

In this paper we began investigations of local rotundity structure of Cesàro–Orlicz sequence spaces ces_φ. It is easy to show that, if σx, where $x \in S(\mathrm{ces}_\varphi)$, is an extreme point (respectively an SU-point) of the Orlicz space l_φ, then x is an extreme point (respectively an SU-point) of ces_φ. Since not all elements $y \in B(l_\varphi)$ can be written as σx for some $x \in B(\mathrm{ces}_\varphi)$, for some Orlicz functions there exists $x \in S(\mathrm{ces}_\varphi)$ such that x is an extreme point (respectively an SU-point) of ces_φ, but

σx is not an extreme point (respectively an SU-point) of the Orlicz space l_φ (see Examples 4 and 5 on p. 419). Therefore, for some Orlicz functions φ, criteria for extreme points and SU-points in Cesàro–Orlicz space ces_φ are essentially weaker than corresponding criteria in Orlicz spaces.

In order to recall criteria for extreme points and SU-points in Orlicz spaces we need to formulate some definitions. We say that $u \in \mathbb{R}$ is a point of strict convexity of φ if $\varphi(u) < \frac{1}{2}(\varphi(u-\varepsilon)+\varphi(u+\varepsilon))$ for any $\varepsilon > 0$. By $\bar{B}_x$ we denote the set of all $n \in \mathbb{N}$ such that φ is affine on the interval $[|x(n)|, |x(n)|+\delta]$ for some $\delta > 0$. Let $\bar{C}_x$ be the set of all $n \in \mathbb{N}$ such that $n \in \operatorname{supp} x$ and φ is affine on the interval $[|x(n)|-\delta, |x(n)|]$ for some $\delta > 0$.

Theorem A. *(See [15, Theorem 1].) An element $x \in S(l_\varphi)$ is an extreme point of $B(l_\varphi)$ if and only if $x(n) = b_\varphi$ for any $n \in N$ or the following conditions are satisfied*:

(i) $\mathrm{I}_\varphi(x) = 1$,
(ii) $|x(n)| \geqslant a_\varphi$ *for all* $n \in N$,
(iii) *at most one number* $x(n)$ *is not a point of strict convexity of* φ.

Theorem B. *(See [8, Theorem 5].) An element $x \in S(l_\varphi)$ is an SU-point of $B(l_\varphi)$ if and only if the following conditions are satisfied*:

(i) $\mathrm{I}_\varphi(\frac{x}{\beta}) < \infty$ *for some* $\beta \in (0,1)$ *or* ($\mathrm{I}_\varphi(x) = 1$ *whenever* $|x| = b_\varphi e_n$ *for some* $n \in \mathbb{N}$),
(ii) $a_\varphi = 0$,
(iii) *if the sets* $\bar{B}_x$ *and* $\bar{C}_x$ *are nonempty, then they are equal and both are singletons.*

2. Results

We start with a simple observation which shows together with Remark 1 that the notion of SU-point is important.

Note 1. Let X be a Banach space, A be its bounded closed convex and nonempty subset and $x \in X \setminus A$. Assume that there exists $y \in A$ such that $y \in P_A(x) := \{z \in A\colon \|x-z\| = d(x,A)\}$ ($d(x,A) := \inf\{\|x-z\|\colon z \in A\}$) and $x-y$ is an SU-point of $B_X(\theta, d(x,A))$. Then $P_A(x) = \{y\}$.

Proof. Assume that there exists $z \in P_A(x)$. Since the set A is convex we have $(y+z)/2 \in A$, hence

$$d(x,A) \leqslant \left\| x - \frac{y+z}{2} \right\| = \left\| \frac{(x-y)+(x-z)}{2} \right\| \leqslant \frac{1}{2}\big\{\|x-y\| + \|x-z\|\big\} = d(x,A).$$

Therefore, we get $\|(x-y)+(x-z)\| = 2d(x,A) = 2\|x-y\|$. Consequently $x-y = x-z$, whence $y = z$. $\square$

Remark 1. Observe that the above note is not true if we assume that $x-y$ is only an extreme point. Namely, let $X = l^2_\infty$ (two-dimensional l_∞ space), $A = B(l^2_\infty)$, $x = (0,2)$. We have $d(x,A) = 1$, $B(\theta, d(x,A)) = B(l^2_\infty)$, $y = (1,1) \in P_A(x)$, $x-y = (-1,1)$ is an extreme point of $B(\theta, d(x,A)) = B(l^2_\infty)$ and $P_A(x) = \{(c,1)\colon c \in [-1,1]\}$.

Let us start now with our results concerning Cesàro–Orlicz sequence spaces. Let A_x be the set of all $n \in \operatorname{supp} x$ such that $\operatorname{supp} x \cap \{n+1, n+2, \ldots\} \neq \emptyset$ and the numbers $\sigma x(n), \sigma x(n+1), \ldots, \sigma x(m-1)$ are inside of the affine intervals of φ, where $m \in \mathbb{N}$ is the smallest number such that $m > n$ and $m \in \operatorname{supp} x$.

Theorem 1. *An element $x \in S(\mathrm{ces}_\varphi)$ is an extreme point of $B(\mathrm{ces}_\varphi)$ if and only if $\mu(\operatorname{supp} x) = \infty$ and $\operatorname{supp} x = \{n\colon \sigma x(n) = b_\varphi\}$ or the following conditions are satisfied*:

(i) $\varrho_\varphi(x) = 1$,
(ii) *if* $a_\varphi > 0$, *then* $\mu(\operatorname{supp} x) = \infty$ *and* $\sigma x(n) \geqslant a_\varphi$ *for all* $n \in \operatorname{supp} x$,
(iii) *at most one number belongs to the set* A_x.

Remark 2. Observe that $x \in l^0$ such that $\mu(\operatorname{supp} x) = \infty$ and $\operatorname{supp} x = \{n\colon \sigma x(n) = b_\varphi\}$ belongs to $S(\mathrm{ces}_\varphi)$ if and only if $a_\varphi = b_\varphi$ since thanks the left-hand side continuity of φ on $\mathbb{R}_+$ we have then that $\varphi(b_\varphi) = 0$.

Proof of Theorem 1. *Sufficiency.* First we will show that if $x \in S(\mathrm{ces}_\varphi)$, $\operatorname{supp} x = \{n\colon \sigma x(n) = b_\varphi\}$ and $\mu(\operatorname{supp} x) = \infty$, then x is an extreme point of $B(\mathrm{ces}_\varphi)$. Assume that there exist $y, z \in S(\mathrm{ces}_\varphi)$ such that $x = \frac{y+z}{2}$ and $y \neq z$. Let m be the smallest natural number such that $y(m) \neq z(m)$. We will consider two cases.

Let $m \in \operatorname{supp} x$. Then $\sigma x(m) = b_\varphi$. Without loss of generality, we may assume that $|x(m)| < |y(m)|$. Therefore

$$\sigma y(m) = \frac{|y(1)| + |y(2)| + \cdots + |y(m-1)| + |y(m)|}{m} = \frac{|x(1)| + |x(2)| + \cdots + |x(m-1)| + |y(m)|}{m}$$
$$> \frac{|x(1)| + |x(2)| + \cdots + |x(m-1)| + |x(m)|}{m} = \sigma x(m) = b_\varphi.$$

Hence $\varrho_\varphi(y) = \infty$, which contradicts the condition $y \in S(\mathrm{ces}_\varphi)$.

Let $m \notin \operatorname{supp} x$. Then, we have $|y(m)| = |z(m)| > 0$ and $\operatorname{sgn} y(m) = -\operatorname{sgn} z(m)$. Let us denote by l the smallest number in $\operatorname{supp} x$ such that $l > m$. Without loss of generality, we may assume that $|x(l)| \leqslant |y(l)|$. Simultaneously, we have $0 = |x(n)| \leqslant |y(n)|$ for $n = m+1, \ldots, l-1$. Therefore

$$\sigma y(l) = \frac{|y(1)| + \cdots + |y(m-1)| + |y(m)| + |y(m+1)| + \cdots + |y(l)|}{l}$$
$$= \frac{|x(1)| + \cdots + |x(m-1)| + |y(m)| + |y(m+1)| + \cdots + |y(l)|}{l}$$
$$\geqslant \frac{|x(1)| + \cdots + |x(m-1)| + |y(m)| + |x(m+1)| + \cdots + |x(l)|}{l} > \sigma x(l) = b_\varphi.$$

Hence $\varrho_\varphi(y) = \infty$ and it is again a contradiction with the assumption $y \in S(\mathrm{ces}_\varphi)$.

Now, we assume that $x \in S(\mathrm{ces}_\varphi)$ satisfies conditions (i)–(iii) of the theorem and there exist $y, z \in S(\mathrm{ces}_\varphi)$ such that $x = \frac{y+z}{2}$ and $y \neq z$. Since ϱ_φ is a convex modular, we have

$$1 = \varrho_\varphi(x) = \varrho_\varphi\left(\frac{y+z}{2}\right) \leqslant \frac{\varrho_\varphi(y) + \varrho_\varphi(z)}{2} \leqslant \frac{1+1}{2} = 1.$$

Then $\varrho_\varphi(y) = \varrho_\varphi(z) = 1$ and $\varrho_\varphi(\frac{y+z}{2}) = \frac{\varrho_\varphi(y)+\varrho_\varphi(z)}{2}$, i.e.

$$\sum_{n=1}^\infty \varphi\left(\frac{1}{n}\sum_{i=1}^n \left|\frac{y(i)+z(i)}{2}\right|\right) = \sum_{n=1}^\infty \varphi\left(\frac{1}{2}\frac{1}{n}\sum_{i=1}^n |y(i)| + \frac{1}{2}\frac{1}{n}\sum_{i=1}^n |z(i)|\right) = \sum_{n=1}^\infty \left(\frac{1}{2}\varphi\left(\frac{1}{n}\sum_{i=1}^n |y(i)|\right) + \frac{1}{2}\varphi\left(\frac{1}{n}\sum_{i=1}^n |z(i)|\right)\right)$$
$$= \frac{1}{2}\left(\sum_{n=1}^\infty \varphi\left(\frac{1}{n}\sum_{i=1}^n |y(i)|\right) + \sum_{n=1}^\infty \varphi\left(\frac{1}{n}\sum_{i=1}^n |z(i)|\right)\right).$$

Thus, for each $n \in \mathbb{N}$, we have

$$\varphi\left(\frac{1}{n}\sum_{i=1}^n \left|\frac{y(i)+z(i)}{2}\right|\right) = \frac{1}{2}\varphi\left(\frac{1}{n}\sum_{i=1}^n |y(i)|\right) + \frac{1}{2}\varphi\left(\frac{1}{n}\sum_{i=1}^n |z(i)|\right). \tag{1}$$

Now, we will show that $x(n) = y(n) = z(n) = 0$ for any $n \notin \operatorname{supp} x$. Note that if $y(n) \neq 0$ for some $n \notin \operatorname{supp} x$, then by the equality $0 = x(n) = \frac{y(n)+z(n)}{2}$, we have $y(n) = -z(n)$ and $|y(n)| = |z(n)| > 0$. Hence $0 = |x(n)| < \frac{1}{2}(|y(n)| + |z(n)|)$, which gives

$$\sigma x(m) = \frac{|x(1)| + \cdots + |x(n)| + \cdots + |x(m)|}{m} = \frac{|\frac{y(1)+z(1)}{2}| + \cdots + |\frac{y(n)+z(n)}{2}| + \cdots + |\frac{y(m)+z(m)}{2}|}{m}$$
$$< \frac{1}{2}\left(\frac{|y(1)| + \cdots + |y(n)| + \cdots + |y(m)|}{m} + \frac{|z(1)| + \cdots + |z(n)| + \cdots + |z(m)|}{m}\right) = \frac{1}{2}\big(\sigma y(m) + \sigma z(m)\big) \tag{2}$$

for each $m \geqslant n$. By condition (ii), there exists $m > n$ such that $\sigma x(m) \geqslant a_\varphi$. Hence

$$\varphi\big(\sigma x(m)\big) < \varphi\left(\frac{\sigma y(m) + \sigma z(m)}{2}\right) \leqslant \frac{\varphi(\sigma y(m)) + \varphi(\sigma z(m))}{2},$$

which contradicts equality (1). So, $n \notin \operatorname{supp} x$ implies that $x(n) = y(n) = z(n) = 0$.

Now, we will show that $|y(n) + z(n)| = |y(n)| + |z(n)|$ for any $n \in \operatorname{supp} x$. Assume that $|y(n) + z(n)| < |y(n)| + |z(n)|$ for some $n \in \operatorname{supp} x$. Then repeating the equalities and the inequality from (2), we get $\sigma x(n) < \frac{1}{2}(\sigma y(n) + \sigma z(n))$. Hence, by condition (ii), we have $\varphi(\sigma x(n)) < \frac{1}{2}(\varphi(\sigma y(n)) + \varphi(\sigma z(n)))$. It is again a contradiction with (1). Therefore, $\min(y(n), z(n)) \geqslant 0$ or $\max(y(n), z(n)) \leqslant 0$.

Let $n_1 \in \mathbb{N}$ be the smallest number such that $y(n_1) \neq z(n_1)$. By the previous part of the proof, we have $n_1 \in \operatorname{supp} x$. Without loss of generality, we may assume that $0 \leqslant |y(n_1)| < |x(n_1)| < |z(n_1)|$. Hence

$$\varphi\big(\sigma y(n_1)\big) < \varphi\big(\sigma z(n_1)\big). \tag{3}$$

Let $m_1 \in \mathbb{N}$ be the smallest number such that $m_1 > n_1$ and $m_1 \in \operatorname{supp} x$ (if $x(n) = 0$ for every $n > n_1$, then by (3), we get the contradiction $\varrho_\varphi(y) < \varrho_\varphi(z)$). For $i \in \{n_1, n_1+1, \ldots, m_1-1\}$, we have $\sigma y(i) < \sigma z(i)$. If there exists $i \in \{n_1, n_1+1, \ldots, m_1-1\}$ such that

$$\varphi(\sigma x(i)) < \frac{1}{2}\big(\varphi(\sigma y(i)) + \varphi(\sigma z(i))\big),$$

then we get a contradiction with (1). Assume now that

$$\varphi(\sigma x(i)) = \frac{1}{2}\big(\varphi(\sigma y(i)) + \varphi(\sigma z(i))\big)$$

for every $i \in \{n_1, n_1+1, \ldots, m_1-1\}$. Then $n_1 \in A_x$. Since $\varrho_\varphi(y) = \varrho_\varphi(z) = 1$, so by (3), there exists n ($n \geqslant m_1$) such that $\sigma z(n) < \sigma y(n)$. Let $n_2 \in \mathbb{N}$ be the smallest number such that $\sigma z(n_2) < \sigma y(n_2)$. Then $|z(n_2)| < |y(n_2)|$, so $n_2 \in \operatorname{supp} x$. If $x(m) = 0$ for every $m > n_2$, then, by condition (ii), we have $a_\varphi = 0$. Proceeding analogously as in the proof of the theorem about rotundity of $(\mathrm{ces}^2_\varphi, \|\cdot\|_\varphi)$ in [14], we get a contradiction with (1).

Now, let us denote by m_2 the smallest number in $\operatorname{supp} x$ such that $m_2 > n_2$. Since $n_2 \notin A_x$, there exists $i \in \{n_2, n_2+1, \ldots, m_2-1\}$ such that

$$\varphi(\sigma x(i)) < \frac{1}{2}\big(\varphi(\sigma y(i)) + \varphi(\sigma z(i))\big),$$

which contradicts equality (1).

Necessity. At first, we assume that $\varrho_\varphi(x) < 1$ and $\mu(\operatorname{supp} x) < \infty$. Let us denote by n the biggest number in $\operatorname{supp} x$ and let $s := |x(1)| + \cdots + |x(n)|$. Since $\varrho_\varphi(x) < 1$, so $\varphi(\frac{s}{n}) < \infty$ and, in consequence, $\frac{s}{n} \leqslant b_\varphi$. Hence, there exists $\delta > 0$ such that $\frac{s+\delta}{n+1} < b_\varphi$. By Theorem 2.1 in [9], we have

$$\sum_{i=n+1}^{\infty} \varphi\left(\frac{s+\delta}{i}\right) < \infty.$$

Hence, remembering that $\varrho_\varphi(x) < 1$, we may assume, without loss of generality, that

$$\sum_{i=1}^{n} \varphi(\sigma x(i)) + \sum_{i=n+1}^{\infty} \varphi\left(\frac{s+\delta}{i}\right) \leqslant 1. \tag{4}$$

Define two sequences y and z, by

$$y = \big(x(1), x(2), \ldots, x(n), \delta, 0, \ldots\big),$$
$$z = \big(x(1), x(2), \ldots, x(n), -\delta, 0, \ldots\big).$$

By (4), we have $\varrho_\varphi(y) = \varrho_\varphi(z) \leqslant 1$. Since $\varrho_\varphi(\lambda y) = \varrho_\varphi(\lambda z) \geqslant \varrho_\varphi(\lambda x) = \infty$ for all $\lambda > 1$, we have $\|y\| = \|z\| = 1$.

Now, let $\varrho_\varphi(x) < 1$, $\mu(\operatorname{supp} x) = \infty$ and there exists $n \in \operatorname{supp} x$ such that $\sigma x(n) < b_\varphi$. Let us denote by m the smallest natural number greater than n such that $x(m) \neq 0$. As above, let $s := |x(1)| + \cdots + |x(n)|$ and let $0 < \delta < \min(|x(n)|, |x(m)|)$ be such a number that $\frac{s+\delta}{n} < b_\varphi$ and

$$\sum_{i=n}^{m-1} \varphi\left(\frac{s+\delta}{i}\right) - \sum_{i=n}^{m-1} \varphi(\sigma x(i)) \leqslant 1 - \varrho_\varphi(x). \tag{5}$$

Define the following sequences:

$$y = \big(x(1), \ldots, x(n) + \operatorname{sgn} x(n)\delta, \ldots, x(m) - \operatorname{sgn} x(m)\delta, \ldots\big),$$
$$z = \big(x(1), \ldots, x(n) - \operatorname{sgn} x(n)\delta, \ldots, x(m) + \operatorname{sgn} x(m)\delta, \ldots\big).$$

By (5) we have $\varrho_\varphi(y) \leqslant 1$. Since $\varrho_\varphi(\lambda y) \geqslant \varrho_\varphi(\lambda x) = \infty$ for any $\lambda > 1$, so $\|y\| = 1$. For z we have $\varrho_\varphi(z) \leqslant \varrho_\varphi(x) < 1$. Assume that there exists $\lambda_0 > 1$ such that $\varrho_\varphi(\lambda_0 z) < \infty$. Then, for any $\lambda \in (1, \lambda_0)$ we have $\varrho_\varphi(\lambda z) < \infty$. Take $\lambda \in (1, \lambda_0)$ such that $\lambda\sigma x(n) < b_\varphi$. Then

$$\varrho_\varphi(\lambda x) = \varrho_\varphi(\lambda z) + \sum_{i=n}^{m-1} \varphi(\lambda\sigma x(i)) - \sum_{i=n}^{m-1} \varphi\left(\lambda\frac{s-\delta}{i}\right) < \infty,$$

which is impossible. Therefore $\varrho_\varphi(\lambda z) = \infty$ for any $\lambda > 1$, whence $\|z\| = 1$.

Now, assume that $\varrho_\varphi(x) = 1$, $a_\varphi > 0$ and $\mu(\operatorname{supp} x) < \infty$. Let us denote by n the biggest number in $\operatorname{supp} x$ and let again $s := |x(1)| + \cdots + |x(n)|$. Since $\frac{s+1}{i} \to 0$ as $i \to \infty$, there exists $m > n$ such that $\frac{s+1}{m} \leqslant a_\varphi$. Define

$$y = \big(\underbrace{x(1), \ldots, x(n), 0, \ldots, 0}_{m-1 \text{ times}}, 1, 0, \ldots\big) \quad \text{and} \quad z = \big(\underbrace{x(1), \ldots, x(n), 0, \ldots, 0}_{m-1 \text{ times}}, -1, 0, \ldots\big).$$

Then, we have $\varrho_\varphi(y) = \varrho_\varphi(z) = \varrho_\varphi(x) = 1$.

Let $\mu(\operatorname{supp} x) = \infty$ and there exists $n \in \operatorname{supp} x$ such that $\sigma x(n) < a_\varphi$. Let us denote by m the smallest number in $\operatorname{supp} x$ such that $m > n$. Let us define the following sequences:

$$y = \big(x(1), \ldots, x(n) + \operatorname{sgn} x(n)\delta, \ldots, x(m) - \operatorname{sgn} x(m)\delta, \ldots\big),$$
$$z = \big(x(1), \ldots, x(n) - \operatorname{sgn} x(n)\delta, \ldots, x(m) + \operatorname{sgn} x(m)\delta, \ldots\big),$$

where $\delta < \min\{|x(n)|, |x(m)|\}$ and $(|x(1)| + \cdots + |x(n)| + \delta) \leqslant na_\varphi$. Then $\varrho_\varphi(y) = \varrho_\varphi(z) = \varrho_\varphi(x) = 1$.

Finally, we will show that condition (iii) is necessary. Assume that A_x has at least two elements. We will show that x is not then an extreme point. Let $n_1, n_2 \in A_x$ and $n_1 \neq n_2$. Let us denote by m_1 (m_2) the smallest number in $\operatorname{supp} x$ such that $m_1 > n_1$ ($m_2 > n_2$). Without loss of generality, we may assume that $m_1 \leqslant n_2$. By the assumption, we have that $\sigma x(n_1), \ldots, \sigma x(m_1 - 1)$ and $\sigma x(n_2), \ldots, \sigma x(m_2 - 1)$ are different than zero and they belong to the enterior of the affine intervals of the function φ. Assume that φ is defined on these intervals by the formula $\varphi(u) = a_i u + b_i$, where $i = n_1, \ldots, m_1 - 1, n_2, \ldots, m_2 - 1$ and define

$$\alpha_1 = \frac{a_{n_1}}{n_1} + \frac{a_{n_1+1}}{n_1+1} + \cdots + \frac{a_{m_1-1}}{m_1-1},$$
$$\alpha_2 = \frac{a_{n_2}}{n_2} + \frac{a_{n_2+1}}{n_2+1} + \cdots + \frac{a_{m_2-1}}{m_2-1}.$$

We may assume without loss of generality that $\alpha_1 \leqslant \alpha_2$. Let $\delta_1 > 0$ be such a number that the intervals

$$\big[\sigma x(n_1) - 2\delta_1, \sigma x(n_1) + 2\delta_1\big], \ldots, \big[\sigma x(m_1 - 1) - 2\delta_1, \sigma x(m_1 - 1) + 2\delta_1\big],$$
$$\big[\sigma x(n_2) - 2\delta_1, \sigma x(n_2) + 2\delta_1\big], \ldots, \big[\sigma x(m_2 - 1) - 2\delta_1, \sigma x(m_2 - 1) + 2\delta_1\big],$$

are inside of the affine intervals of the function φ and the inequality

$$2\delta_1 < \min\big\{\big|x(n_1)\big|, \big|x(m_2)\big|, \big|x(n_2)\big|, \big|x(m_2)\big|\big\}$$

is satisfied. Let $\delta_2 := \frac{\alpha_1}{\alpha_2}\delta_1$. It is obvious that $\delta_2 \leqslant \delta_1$. If $m_1 < n_2$, then we define two sequences y and z, by

$$y = \big(x(1), \ldots, x(n_1) + \operatorname{sgn} x(n_1)\delta_1, \ldots, x(m_1) - \operatorname{sgn} x(m_1)\delta_1, \ldots, x(n_2) - \operatorname{sgn} x(n_2)\delta_2, \ldots, x(m_2) + \operatorname{sgn} x(m_2)\delta_2, \ldots\big),$$
$$z = \big(x(1), \ldots, x(n_1) - \operatorname{sgn} x(n_1)\delta_1, \ldots, x(m_1) + \operatorname{sgn} x(m_1)\delta_1, \ldots, x(n_2) + \operatorname{sgn} x(n_2)\delta_2, \ldots, x(m_2) - \operatorname{sgn} x(m_2)\delta_2, \ldots\big).$$

In the case, when $m_1 = n_2$, we define y and z similarly, namely

$$y(m_1) := x(m_1) - \operatorname{sgn} x(m_1)(\delta_1 + \delta_2)$$

and

$$z(m_1) := x(m_1) + \operatorname{sgn} x(m_1)(\delta_1 + \delta_2).$$

For the sequences y and z, we have

$$\sum_{i=n_1}^{m_1-1} \varphi\big(\sigma y(i)\big) = \sum_{i=n_1}^{m_1-1} \varphi\big(\sigma x(i)\big) + \alpha_1\delta_1, \qquad \sum_{i=n_2}^{m_2-1} \varphi\big(\sigma y(i)\big) = \sum_{i=n_2}^{m_2-1} \varphi\big(\sigma x(i)\big) - \alpha_2\delta_2,$$
$$\sum_{i=n_1}^{m_1-1} \varphi\big(\sigma z(i)\big) = \sum_{i=n_1}^{m_1-1} \varphi\big(\sigma x(i)\big) - \alpha_1\delta_1, \qquad \sum_{i=n_2}^{m_2-1} \varphi\big(\sigma z(i)\big) = \sum_{i=n_2}^{m_2-1} \varphi\big(\sigma x(i)\big) + \alpha_2\delta_2.$$

Hence $\varrho_\varphi(y) = \varrho_\varphi(z) = \varrho_\varphi(x) = 1$. $\square$

Let us denote by B_x the set of all $n \in \mathbb{N}$ such that $n + 1 \in \operatorname{supp} x$ and φ is affine on the interval $[\sigma x(n), \sigma x(n) + \delta]$ for some $\delta > 0$.

Let C_x be the set of all $n \in \mathbb{N}$ such that $n \in \operatorname{supp} x$ and φ is affine on the interval $[\sigma x(n) - \delta, \sigma x(n)]$ for some $\delta > 0$.

Theorem 2. *An element $x \in S(\mathrm{ces}_\varphi)$ is an SU-point of $B(\mathrm{ces}_\varphi)$ if and only if the following conditions are satisfied*:

(i) $\rho_\varphi(\frac{x}{\beta}) < \infty$ *for some $\beta \in (0, 1)$ or ($\varrho_\varphi(x) = 1$ whenever $|x| = nb_\varphi e_n$ for some $n \in \mathbb{N}$),*
(ii) $a_\varphi = 0$,
(iii) *if the sets B_x and C_x are nonempty, then they are equal and both are singletons.*

Proof. *Sufficiency.* Let $x \in S(\mathrm{ces}_\varphi)$ satisfy the conditions of the theorem. We will show that x is an SU-point. Let $y \in S(\mathrm{ces}_\varphi)$ and $x \neq y$. First, we will show that $\varrho_\varphi(\frac{x+y}{2}) < 1$. We notice that it is enough to show that, for some $n \in \mathbb{N}$, we have

$$\varphi\left(\frac{1}{n}\sum_{i=1}^{n}\left|\frac{x(i)+y(i)}{2}\right|\right) < \frac{1}{2}\left(\varphi\left(\frac{1}{n}\sum_{i=1}^{n}\left|x(i)\right|\right) + \varphi\left(\frac{1}{n}\sum_{i=1}^{n}\left|y(i)\right|\right)\right). \tag{6}$$

Let k be the smallest natural number such that $x(k) \neq y(k)$. If $|x(k)| = |y(k)| > 0$, then $x(k) = -y(k)$. Hence

$$\begin{aligned}\sigma\frac{x+y}{2}(k) &= \frac{|\frac{x(1)+y(1)}{2}| + \cdots + |\frac{x(k)+y(k)}{2}|}{k} < \frac{1}{2}\left(\frac{|x(1)| + \cdots + |x(k)|}{k}\right) + \frac{1}{2}\left(\frac{|y(1)| + \cdots + |y(k)|}{k}\right)\\ &= \frac{1}{2}\sigma x(k) + \frac{1}{2}\sigma y(k).\end{aligned}$$

Since $a_\varphi = 0$, we have

$$\varphi\left(\sigma\frac{x+y}{2}(k)\right) < \varphi\left(\frac{1}{2}\sigma x(k) + \frac{1}{2}\sigma y(k)\right) \leqslant \frac{1}{2}\left[\varphi\left(\sigma x(k)\right) + \varphi\left(\sigma y(k)\right)\right],$$

which means that (6) is satisfied for $n = k$.

Now, let $|x(k)| \neq |y(k)|$. Without loss of generality, we may assume that $|y(k)| < |x(k)|$. Then, we have

$$\frac{1}{k}\sum_{i=1}^{k}\left|y(i)\right| < \frac{1}{k}\sum_{i=1}^{k}\left|x(i)\right|.$$

Since $a_\varphi = 0$, so

$$\varphi\left(\frac{1}{k}\sum_{i=1}^{k}\left|y(i)\right|\right) < \varphi\left(\frac{1}{k}\sum_{i=1}^{k}\left|x(i)\right|\right).$$

If the inequality

$$\varphi\left(\frac{1}{n}\sum_{i=1}^{n}\left|y(i)\right|\right) \leqslant \varphi\left(\frac{1}{n}\sum_{i=1}^{n}\left|x(i)\right|\right)$$

were satisfied for others $n \in \mathbb{N}$, then we would immediately have

$$\varrho_\varphi\left(\frac{x+y}{2}\right) \leqslant \frac{1}{2}\left(\varrho_\varphi(x) + \varrho_\varphi(y)\right) < 1,$$

and the proof would be finished. Therefore assume that there exists $n \in \mathbb{N}$ such that

$$\varphi\left(\frac{1}{n}\sum_{i=1}^{n}\left|x(i)\right|\right) < \varphi\left(\frac{1}{n}\sum_{i=1}^{n}\left|y(i)\right|\right), \tag{7}$$

whence

$$\frac{1}{n}\sum_{i=1}^{n}\left|x(i)\right| < \frac{1}{n}\sum_{i=1}^{n}\left|y(i)\right|.$$

Let m be the smallest natural number which satisfies (7). We have $k < m$ and $|x(m)| < |y(m)|$. If inequality (6) is not satisfied for k, then $k \in C_x$. Hence $m \notin B_x$. If $x(m+1) \neq 0$, then we get inequality (6) for m. If there exists $p > m$ such that $x(m+1) = \cdots = x(p) = 0$ and $x(p+1) \neq 0$ then, by $p \notin B_x$, we get (6) on a pth coordinate. If we have $x(n) = 0$ for every $n > m$, then the proof can proceed analogously as in the theorem about rotundity of $(\mathrm{ces}^2_\varphi, \|\cdot\|_\varphi)$ in [14]. Hence $\varrho_\varphi(\frac{x+y}{2}) < 1$.

Now, we must only show that $\|\frac{x+y}{2}\| < 1$, which is true if and only if $\varrho_\varphi((1+\epsilon)\frac{x+y}{2}) < \infty$ for some $\epsilon > 0$. Let $|x| = nb_\varphi e_n$ and $\varrho_\varphi(x) = 1$. Since $y \neq x$, then $|y(n)| < |x(n)| = nb_\varphi$, $\sigma y(m) \leqslant b_\varphi$ for $m = 1, 2, \ldots, n-1$ and $\sigma y(m) < b_\varphi$ for $m \geqslant n$. Therefore, for any $\epsilon_1 \in (0, \frac{b_\varphi - \sigma y(n)}{b_\varphi + \sigma y(n)})$, we have

$$\begin{aligned}\sum_{i=1}^{n}\varphi\left(\sigma(1+\epsilon_1)\frac{x+y}{2}(i)\right) &= \sum_{i=1}^{n-1}\varphi\left(\sigma(1+\epsilon_1)\frac{y}{2}(i)\right) + \varphi\left(\sigma(1+\epsilon_1)\frac{x+y}{2}(n)\right)\\ &\leqslant \sum_{i=1}^{n-1}\varphi\left(\frac{1+\epsilon_1}{2}\sigma y(i)\right) + \varphi\left(\frac{1+\epsilon_1}{2}\left(\sigma y(n) + b_\varphi\right)\right)\end{aligned}$$

$$< \sum_{i=1}^{n-1} \varphi\left(\frac{1+\epsilon_1}{2}\sigma y(i)\right) + \varphi(b_\varphi) < \infty.$$

We know that $\sigma x(m) \leqslant \sigma x(n+1) = \frac{nb_\varphi}{n+1}$ for any $m \neq n$. Let $\epsilon_2 \in (0, \frac{1}{2n+1})$ and $a = \frac{1-\epsilon_2}{1+\epsilon_2}$. We have $\frac{n}{n+1} < a < 1$ and $\frac{1+\epsilon_2}{2}a + \frac{1+\epsilon_2}{2} = 1$. Hence

$$\sum_{i=n+1}^{\infty} \varphi\left(\sigma(1+\epsilon_2)\frac{x+y}{2}(i)\right) \leqslant \sum_{i=n+1}^{\infty} \varphi\left(\frac{1+\epsilon_2}{2}\sigma x(i) + \frac{1+\epsilon_2}{2}\sigma y(i)\right) = \sum_{i=n+1}^{\infty} \varphi\left(\frac{1+\epsilon_2}{2}a \cdot \sigma\frac{x}{a}(i) + \frac{1+\epsilon_2}{2}\sigma y(i)\right)$$

$$\leqslant \frac{1+\epsilon_2}{2}a \sum_{i=n+1}^{\infty} \varphi\left(\sigma\frac{x}{a}(i)\right) + \frac{1+\epsilon_2}{2}\sum_{i=n+1}^{\infty} \varphi\big(\sigma y(i)\big) < \infty,$$

because the convergence of the series $\sum_{i=n+1}^{\infty} \varphi(\sigma \frac{x}{a}(i))$ follows from Theorem 2.1 in [9] for $k = \frac{nb_\varphi}{a}$ and from the fact that $\sigma\frac{x(i)}{a} \leqslant b_\varphi$ for any $i \geqslant n+1$. Therefore, taking $\epsilon \in (0, \min\{\frac{b_\varphi - \sigma y(n)}{b_\varphi + \sigma y(n)}, \frac{1}{2n+1}\})$, we get $\varrho_\varphi((1+\epsilon)\frac{x+y}{2}) < \infty$, whence $\|\frac{x+y}{2}\| < 1$.

Now, let $\varrho_\varphi(\frac{x}{\beta}) < \infty$ for some $\beta \in (0,1)$. Then, we have $\epsilon \in (0,1)$, $\beta = \frac{1-\epsilon}{1+\epsilon}$ and $\frac{1+\epsilon}{2}\beta + \frac{1+\epsilon}{2} = 1$ for $\epsilon = \frac{1-\beta}{1+\beta}$. Therefore

$$\varrho_\varphi\left((1+\epsilon)\frac{x+y}{2}\right) = \varrho_\varphi\left(\frac{1+\epsilon}{2}\beta\frac{x}{\beta} + \frac{1+\epsilon}{2}y\right) \leqslant \frac{1+\epsilon}{2}\varrho_\varphi\left(\frac{x}{\beta}\right) + \frac{1+\epsilon}{2}\varrho_\varphi(y) < \infty,$$

whence we get again $\|\frac{x+y}{2}\| < 1$. By the arbitrariness of y from $S(\mathrm{ces}_\varphi)$, x is an SU-point.

Necessity. Assume that $x \in S(\mathrm{ces}_\varphi)$ is an SU-point and x does not satisfy condition (i), which means that $\varrho_\varphi(\frac{x}{\beta}) = \infty$ for every $\beta \in (0,1)$ and x is not of the form $|x| = nb_\varphi e_n$. Since x is an extreme point, by Theorem 1, we have $\varrho_\varphi(x) = 1$.

First, we assume that there exists $i \in \mathbb{N}$ such that $\sigma x(i) = b_\varphi$. Take $m \in \operatorname{supp} x$ such that $m \neq i$ and define

$$y(n) = \begin{cases} \frac{x(m)}{2} & \text{for } n = m, \\ x(m+1) + \operatorname{sgn} x(m+1)\frac{x(m)}{2} & \text{for } n = m+1, \\ x(n) & \text{for } n \in N \setminus \{m, m+1\}. \end{cases}$$

We have $\sigma y(m) < \sigma\frac{x+y}{2}(m) < \sigma x(m)$ and $\sigma y(n) = \sigma\frac{x+y}{2}(n) = \sigma x(n)$ for the others n. Hence $\varrho_\varphi(y) \leqslant \varrho_\varphi((x+y)/2) \leqslant 1$ and $\varrho_\varphi(\frac{y}{\beta}) = \varrho_\varphi(\frac{x+y}{2\beta}) = \infty$ for every $\beta \in (0,1)$, whence $\|y\| = \|\frac{x+y}{2}\| = 1$, so x is not an SU-point.

Now, let $\sigma x(n) < b_\varphi$ for every $n \in \mathbb{N}$ and m be the smallest number in $\operatorname{supp} x$. Consider the sequence y defined above. We have $\varrho_\varphi(y) \leqslant \varrho_\varphi(\frac{x+y}{2}) \leqslant 1$. Since $\varrho_\varphi(\frac{x}{\beta}) = \infty$ for every $\beta \in (0,1)$, $\sigma\frac{x}{\beta}(i) = 0$ for $i = 1, 2, \ldots, m-1$ and $\varphi(\sigma\frac{x}{\beta}(m)) < \infty$ for $\beta \in (\frac{\sigma x(m)}{b_\varphi}, 1)$, so

$$\sum_{i=m+1}^{\infty} \varphi\left(\sigma\frac{y}{\beta}(i)\right) = \sum_{i=m+1}^{\infty} \varphi\left(\sigma\frac{x+y}{2\beta}(i)\right) = \sum_{i=m+1}^{\infty} \varphi\left(\sigma\frac{x}{\beta}(i)\right) = \infty.$$

Hence $\varrho_\varphi(y) = \varrho_\varphi(\frac{x+y}{2\beta}) = \infty$, which implies that $\|y\| = \|\frac{x+y}{2}\| = 1$ and consequently, x is not an SU-point.

Now, we assume that condition (i) is satisfied and $a_\varphi > 0$. If x were an SU-point, then it would also be an extreme point. Therefore, by Theorem 1, we would get $\mu(\operatorname{supp} x) = \infty$ and $\sigma x(n) \geqslant a_\varphi$ for any $n \in \operatorname{supp} x$. Let $\beta \in (0,1)$. Then $\frac{\sigma x(n)}{\beta} \geqslant \frac{a_\varphi}{\beta}$ for any $n \in \operatorname{supp} x$. Hence $\varphi(\frac{\sigma x(n)}{\beta}) \geqslant \varphi(\frac{a_\varphi}{\beta}) > 0$ for any $n \in \operatorname{supp} x$, that is, $\varrho_\varphi(\frac{x}{\beta}) = \infty$ for any $\beta \in (0,1)$, which is a contradiction. Hence $a_\varphi = 0$.

Let us show the necessity of condition (iii). Let us assume that there exist $n \in B_x$ and $m \in C_x$ such that $n \neq m$. We will show that x is not an SU-point. Since $n \in B_x$, so $n+1 \in \operatorname{supp} x$ and there exists η_1 such that the function φ is affine on the interval $[\sigma x(n), \sigma x(n) + \eta_1]$. By the assumption that $m \in C_x$, we know that $m \in \operatorname{supp} x$ and there exists η_2 such that φ is affine on the interval $[\sigma x(m) - \eta_2, \sigma x(m)]$. Let the function φ be defined by the following formulas: $\varphi(u) = a_n u + b_n$ on the interval $[\sigma x(n), \sigma x(n) + \eta_1]$ and $\varphi(u) = a_m u + b_m$ on the interval $[\sigma x(m) - \eta_2, \sigma x(m)]$. Define $\alpha_1 = \frac{a_n}{n}$ and $\alpha_2 = \frac{a_m}{m}$. Without loss of generality, we may assume that $n < m$ and $\alpha_1 \leqslant \alpha_2$. Let δ_1 satisfy the condition $0 < 2\delta_1 < \min\{|x(n+1)|, |x(m)|, \eta_1, \eta_2\}$ and $\delta_2 := \frac{\alpha_1}{\alpha_2}\delta_1$. For y defined by the formula

$$y = \big(x(1), \ldots, x(n) + \operatorname{sgn} x(n)\delta_1, x(n+1) - \operatorname{sgn} x(n+1)\delta_1, \ldots, x(m) - \operatorname{sgn} x(m)\delta_2, x(m+1) + \operatorname{sgn} x(m+1)\delta_2, \ldots\big)$$

(in the case, when $n+1 = m$, we define $y(m) = x(m) - \operatorname{sgn} x(m)(\delta_1 + \delta_2)$), we have

$$\varrho_\varphi(y) = \sum_{i=1}^{\infty} \varphi\big(\sigma y(i)\big) = \sum_{i=1}^{\infty} \varphi\big(\sigma x(i)\big) + \alpha_1\delta_1 - \alpha_2\delta_2 = \sum_{i=1}^{\infty} \varphi\big(\sigma x(i)\big) = \varrho_\varphi(x) = 1.$$

Analogously we get that $\varrho_\varphi(\frac{x+y}{2}) = 1$. Therefore $\|y\| = \|\frac{x+y}{2}\| = 1$, so x is not an SU-point. $\square$

Theorem 3. *The Cesàro–Orlicz space* ces_φ *has always extreme points.*

Proof. If $a_\varphi = b_\varphi$, then $x = (x_n)_{n=1}^\infty$, where $x_n = b_\varphi$ for any $n \in N$, is an extreme point of $S(\mathrm{ces}_\varphi)$. If $\sum_{i=1}^\infty \varphi(\frac{b_\varphi}{i}) \geqslant 1$, then y defined for all $i \in \mathbb{N}$ by the equalities $\sigma y(i) = \max(\frac{u_0}{i}, a_\varphi)$, where $\sum_{i=1}^\infty \varphi(\frac{u_0}{i}) = 1$, is an extreme point. Finally, assume that $0 < \sum_{i=1}^\infty \varphi(\frac{b_\varphi}{i}) < 1$. We can find $n \in N$ and $u_1 \in [0, b_\varphi)$ such that $n\varphi(b_\varphi) + \sum_{i=n+1}^\infty \varphi(\frac{nb_\varphi + u_1}{i}) = 1$. Then $z = (z_i)_{i=1}^\infty$ with z_i satisfying $\sigma z(i) = b_\varphi$ for $i = 1, \ldots, n$ and $\sigma z(i) = \max(\frac{nb_\varphi + u_1}{i}, a_\varphi)$ for the others $i \in \mathbb{N}$, is an extreme point. $\square$

Remark 3. An analogous theorem to the last one is not true for SU-points. For example, ces_φ generated by an Orlicz function φ vanishing outside zero has not SU-points.

Theorem 4. *The space* $(\mathrm{ces}_\varphi)_a$ *has no extreme points if and only if* $a_\varphi > 0$.

Proof. *Sufficiency.* We notice first that, if $a_\varphi > 0$, then $x \in (\mathrm{ces}_\varphi)_a$ if and only if $\sigma x \in c_0$. Take any $x \in S((\mathrm{ces}_\varphi)_a)$. There exists $k \in N$ such that $\sigma x(n) \leqslant a_\varphi/2$ for any $n > k$. Define $y(k+1) = x(k+1) + \operatorname{sgn} x(k+1)\frac{a_\varphi}{2}$, $z(k+1) = x(k+1) - \operatorname{sgn} x(k+1)\frac{a_\varphi}{2}$ and $y(n) = z(n) = x(n)$ for the others $n \in N$. We get $y \neq z$, $x = \frac{y+z}{2}$ and $\varrho_\varphi(y) = \varrho_\varphi(z) = \varrho_\varphi(x) \leqslant 1$. By convexity of the norm, we have $y, z \in S((\mathrm{ces}_\varphi)_a)$. Therefore, x is not an extreme point.

Necessity. Let $a_\varphi = 0$. Then $\varphi(b_\varphi) > 0$ and the elements y or z which were defined in the proof of Theorem 3 are extreme points of $S((\mathrm{ces}_\varphi)_a)$. $\square$

Remark 4. As it has been noticed in Theorem 4, if we assume that $a_\varphi > 0$, then the space $(\mathrm{ces}_\varphi)_a$ has no extreme points. It is worth noticing that $c_0 \subsetneq (\mathrm{ces}_\varphi)_a$. Namely, we have $\sigma x(i) \to 0$ as $i \to \infty$ for any $x \in c_0$. Hence $c_0 \subset (\mathrm{ces}_\varphi)_a$. We will show that $c_0 \neq (\mathrm{ces}_\varphi)_a$. Let $x(i) = n$ for $i = 10^n$ and $x(i) = 0$ for the others $i \in N$. Since $\sigma x(i) \to 0$ as $i \to \infty$, we have $x \in (\mathrm{ces}_\varphi)_a$ and $x \notin l^\infty$. Therefore, in case when $a_\varphi > 0$, $(\mathrm{ces}_\varphi)_a$ is a Banach sequence space without extreme points in its unit ball, different from c_0.

Let us recall here that Orlicz spaces without extreme points are characterized in [13].

3. Some examples

It is easy to observe that conditions of Theorem 2 imply the corresponding conditions of Theorem 1. We will show that they are stronger, i.e. in some spaces ces_φ we will find examples of extreme points which are not SU-points.

The first example shows that the element $x = (x(n)) \in S(\mathrm{ces}_\varphi)$ such that $\mu(\operatorname{supp} x) = \infty$ and $\operatorname{supp} x = \{n\colon\ \sigma x(n) = b_\varphi\}$, is not an SU-point.

Example 1. Let n_1, n_2 be two smallest natural numbers in $\operatorname{supp} x$ such that $n_1 < n_2$. Put

$$y = \big(\underbrace{0, 0, \ldots, 0}_{n_2-1 \text{ times}}, x(n_2) + \operatorname{sgn} x(n_2) n_1 b_\varphi, x(n_2+1), \ldots\big).$$

The assumptions about x imply that $a_\varphi = b_\varphi$, so $\varrho_\varphi(y) = \varrho_\varphi(\frac{x+y}{2}) = 0$ and $\varrho_\varphi(\lambda y) = \varrho_\varphi(\lambda\frac{x+y}{2}) = \infty$ for $\lambda > 1$. Therefore $\|y\| = \|\frac{x+y}{2}\| = 1$. Since $y \neq x$, x is not an SU-point.

The next example shows that condition (i) from Theorem 2 is essentially stronger than condition (i) from Theorem 1.

Example 2. Let us define the Orlicz function φ by the formula

$$\varphi(u) = \begin{cases} \frac{12}{2\pi^2 - 9} u^2 & \text{for } u \in [0, \frac{1}{2}], \\ \infty & \text{for } u \in (\frac{1}{2}, \infty), \end{cases}$$

and let $x = (\frac{1}{2}, \frac{1}{2}, 0, 0, \ldots)$. We have $\varrho_\varphi(x) = 1$, $a_\varphi = 0$ and $A_x = \emptyset$, so x is an extreme point of $B(\mathrm{ces}_\varphi)$. Since $\varrho_\varphi(\frac{x}{\beta}) = \infty$ for any $\beta \in (0, 1)$ and $\mu(\operatorname{supp} x) = 2$, so x is not an SU-point.

Example 3 shows that condition (iii) from Theorem 1 is essentially weaker than the corresponding condition (iii) from Theorem 2.

Example 3. Let us define φ by the formula

$$\varphi(u) = \begin{cases} \frac{6}{\pi^2}u^2 & \text{for } u \in [0, \frac{1}{3}], \\ \frac{6}{\pi^2}(\frac{37}{12}u - \frac{33}{36}) & \text{for } u \in (\frac{1}{3}, \infty). \end{cases}$$

Then $x = (\frac{2}{3}, 0, \frac{1}{3}, 0, 0, \ldots)$ satisfies conditions of Theorem 1, since $\varrho_\varphi(x) = 1$ ($\varrho_\varphi(\lambda x) < \infty$ for any $\lambda > 0$), $a_\varphi = 0$ and $A_x = \emptyset$. Simultaneously, since $2 \in B_x$ and $1 \in C_x$, so x is not an SU-point.

As we noticed on p. 411, if σx, where $x \in S(\mathrm{ces}_\varphi)$, is an extreme point (respectively an SU-point) of the Orlicz space, then x is an extreme point (respectively an SU-point) of the Cesàro–Orlicz space. The next two examples show that the inverse relations are not true.

Example 4. Let $a_\varphi = b_\varphi = 1$. Then $x = (0, 2, 1, 1, \ldots)$ is an extreme point of $B(\mathrm{ces}_\varphi)$. However $\sigma x = (0, 1, 1, 1, \ldots)$, so it is not an extreme point of $B(l^\varphi)$, since for $y = (1, 1, 1, 1, \ldots)$ and $z = (-1, 1, 1, 1, \ldots)$, we have $y, z \in S(l^\varphi)$ and $\sigma x = \frac{y+z}{2}$.

Example 5. Consider the following Orlicz function:

$$\varphi(u) = \begin{cases} \frac{6}{\pi^2}u^2 & \text{for } u \in [0, \frac{1}{2}], \\ \frac{6}{\pi^2}(\frac{3}{2}u - \frac{1}{2}) & \text{for } u \in (\frac{1}{2}, \infty). \end{cases}$$

It is easy to show that $x = e_1$ is an SU-point of the Cesàro–Orlicz space ces_φ. Defining $y = (\frac{1}{2}, 1, \frac{1}{3}, \ldots, \frac{1}{n}, \ldots)$, we get $\|y\| = \|\frac{y+\sigma x}{2}\| = 1$, which means that σx is not an SU-point of the Orlicz space l^φ.

Acknowledgment

The authors thank the referee for valuable comments and suggestions concerning the redaction of this paper.

References

[1] Programma van Jaarlijkse Prijsvragen (Annual Problem Section), Nieuw Arch. Wiskd. 16 (1968) 47–51.
[2] P. Bandyopadhyay, D. Huang, B.L. Lin, Rotund points, nested sequence of balls and smoothness in Banach spaces, Comment. Math. 44 (2) (2004) 163–186.
[3] G. Bennett, Factorizing the classical inequalities, Mem. Amer. Math. Soc. 120 (576) (1996).
[4] S.T. Chen, Geometry of Orlicz Spaces, Dissertationes Math. (Rozprawy Mat.), vol. 356, Polish Acad. Sci., Warsaw, 1996.
[5] S.T. Chen, Y.A. Cui, H. Hudzik, B. Sims, Geometric properties related to fixed point theory in some Banach function lattices, in: Handbook of Metric Fixed Point Theory, Kluwer Acad. Publ., Dordrecht, 2001, pp. 339–389.
[6] Y.A. Cui, H. Hudzik, Some geometric properties related to fixed point theory in Cesàro sequence spaces, Collect. Math. 50 (3) (1999) 277–288.
[7] Y.A. Cui, H. Hudzik, Packing constant for Cesàro sequence spaces, Nonlinear Anal. 47 (2001) 2695–2702.
[8] Y.A. Cui, H. Hudzik, C. Meng, On some local geometry of Orlicz sequence spaces equipped with the Luxemburg norm, Acta Math. Hungar. 80 (1–2) (1998) 143–154.
[9] Y.A. Cui, H. Hudzik, N. Petrot, S. Suantai, A. Szymaszkiewicz, Basic topological and geometric properties of Cesàro–Orlicz spaces, Proc. Indian Acad. Sci. 115 (4) (2005) 461–476.
[10] Y.A. Cui, L. Jie, R. Płuciennik, Local uniform nonsquareness in Cesàro sequence spaces, Comment. Math. 37 (1997) 47–58.
[11] Y.A. Cui, C. Meng, R. Płuciennik, Banach–Saks property and property (β) in Cesàro sequence spaces, Southeast Asian Bull. Math. 24 (2000) 201–210.
[12] J. Diestel, Sequences and Series in Banach Spaces, Grad. Texts in Math., vol. 92, Springer-Verlag, Berlin, 1984.
[13] P. Foralewski, H. Hudzik, R. Płuciennik, Orlicz spaces without extreme points, submitted for publication.
[14] P. Foralewski, H. Hudzik, A. Szymaszkiewicz, Some remarks on Cesàro–Orlicz sequence spaces, submitted for publication.
[15] R. Grząślewicz, H. Hudzik, W. Kurc, Extreme and exposed points in Orlicz spaces, Canad. J. Math. 44 (3) (1992) 505–515.
[16] A.A. Jagers, A note on Cesàro sequence spaces, Nieuw Arch. Wiskd. 22 (1974) 113–124.
[17] L.V. Kantorovich, G.P. Akilov, Functional Analysis, Nauka, Moscow, 1977 (in Russian).
[18] M.A. Krasnoselskiĭ, Ya.B. Rutickiĭ, Convex Functions and Orlicz Spaces, P. Nordhoff Ltd., Groningen, 1961 (translation from Russian).
[19] P.Y. Lee, Cesàro sequence spaces, Math. Chronicle New Zealand 13 (1984) 29–45.
[20] G.M. Leibowitz, A note on the Cesàro sequence spaces, Tamkang J. Math. 2 (1971) 151–157.
[21] J. Lindenstrauss, L. Tzafriri, Classical Banach Spaces I. Sequence Spaces, Springer-Verlag, Berlin, 1977.
[22] J. Lindenstrauss, L. Tzafriri, Classical Banach Spaces II. Function Spaces, Springer-Verlag, Berlin, 1979.
[23] W.A.J. Luxemburg, Banach function spaces, thesis, Delft, 1955.
[24] L. Maligranda, Orlicz Spaces and Interpolation, Semin. Math., vol. 5, Universidade Estadual de Campinas, Campinas, SP, Brazil, 1989.
[25] L. Maligranda, N. Petrot, S. Suantai, On the James constant and B-convexity of Cesàro and Cesàro–Orlicz sequence spaces, J. Math. Anal. Appl. 326 (2007) 312–331.
[26] J. Musielak, Orlicz Spaces and Modular Spaces, Lecture Notes in Math., vol. 1034, Springer-Verlag, Berlin, 1983.
[27] M.M. Rao, Z.D. Ren, Theory of Orlicz Spaces, Marcel Dekker Inc., New York, 1991.
[28] J.S. Shiue, Cesàro sequence spaces, Tamkang J. Math. 1 (1970) 19–25.